Equilibration & Hydrodynamics vs. Collision Energy

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Outline

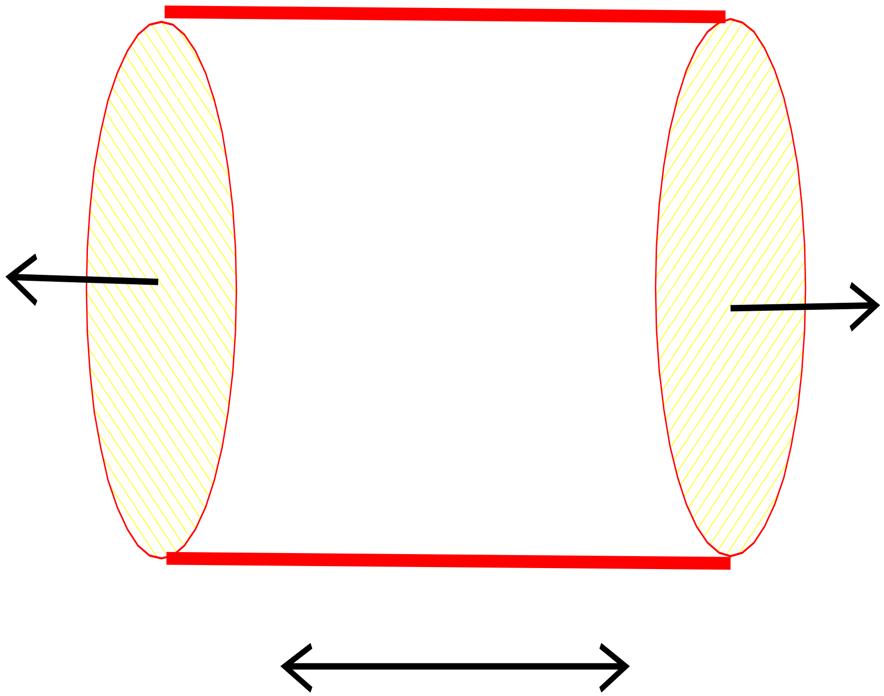
1. Review the Energy Dependence of Ideal Hydro
   - Review, J.P. Blaizot and J.Y. Ollitrault, Nucl. Phys. A458 (1986) 745.
2. Viscosity of Heavy Ion Collisions
   - Two limitting cases for the viscosity:

$$\eta \propto \frac{T}{\sigma_o} \qquad \eta \propto T^3$$

3. Solve viscous 1+1 viscous hydro with radial and Bjorken symmetries.
4. Show the important effects for Heavy Ion Collisions
5. Discuss the energy dependence of the solution.

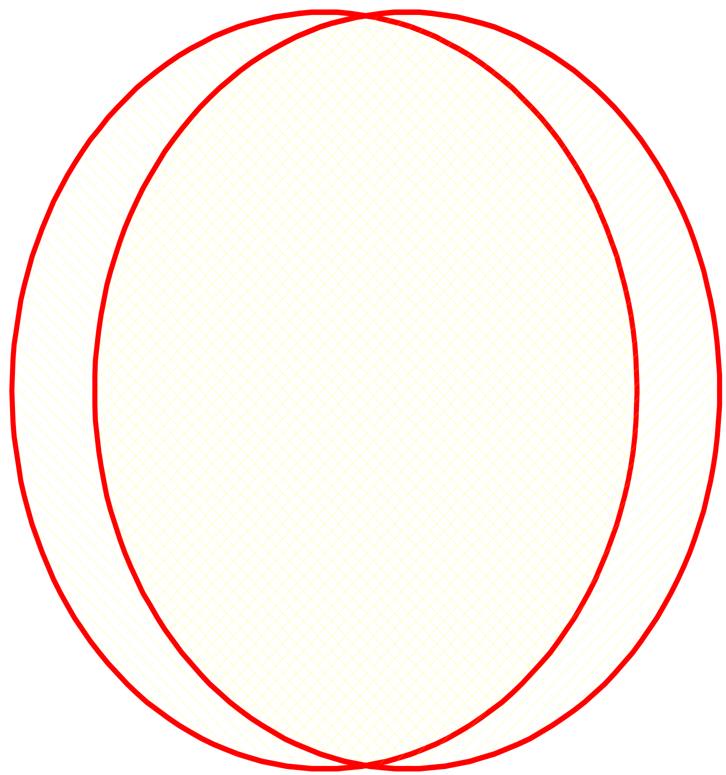
Overview

Bjorken Expansion



beam direction

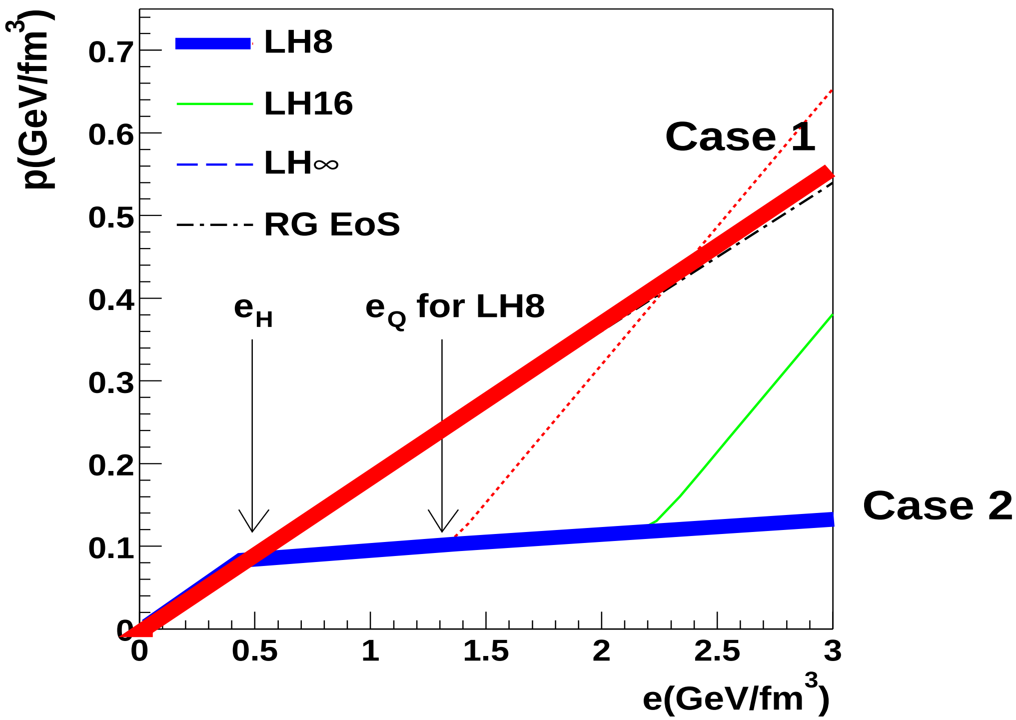
Almost Central -- Head on View



What Happens? What does Ideal Hydro Say?

Ideal Hydrodynamic Response: $\partial_\mu T^{\mu\nu} = 0$

$$(e + p)\,\partial_t v = -\nabla p$$



What is the mean $\langle p_T \rangle$ for these EOS?

Case 1:

1. For massless particles

$$\langle p_T \rangle = \frac{\pi}{4} \times \underbrace{\langle E \rangle}_{\text{Energy per particle}}$$

2. The average energy per particle is

$$\langle E \rangle = \frac{e}{n} = \underbrace{\frac{e}{e+p}}_{3/4} \times \underbrace{\frac{s}{n}}_{\simeq 3.6} \times T$$

$$\langle p_T \rangle \simeq 2.12\, T$$

Case 2:

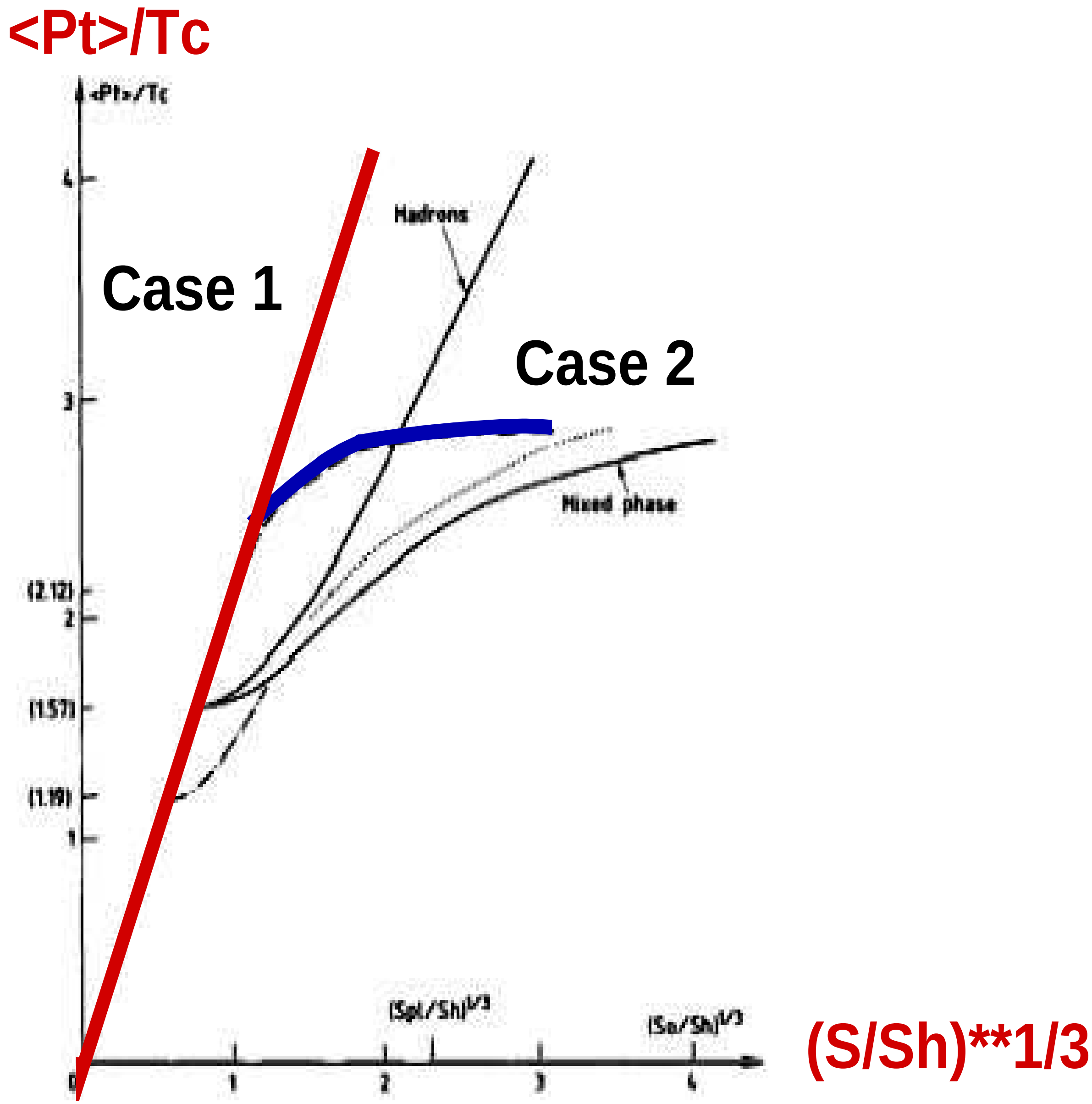
1. For massless particles

$$\langle p_T \rangle = \frac{\pi}{4} \times \underbrace{\langle E \rangle}_{\text{Energy per particle}}$$

2. With $p = p_c$ and $T = T_c$ and $s, e \to \infty$

$$\langle E \rangle = \frac{e}{n} = \underbrace{\frac{e}{e+p}}_{\simeq 1} \times \underbrace{\frac{s}{n}}_{\simeq 3.6} \times T_c$$

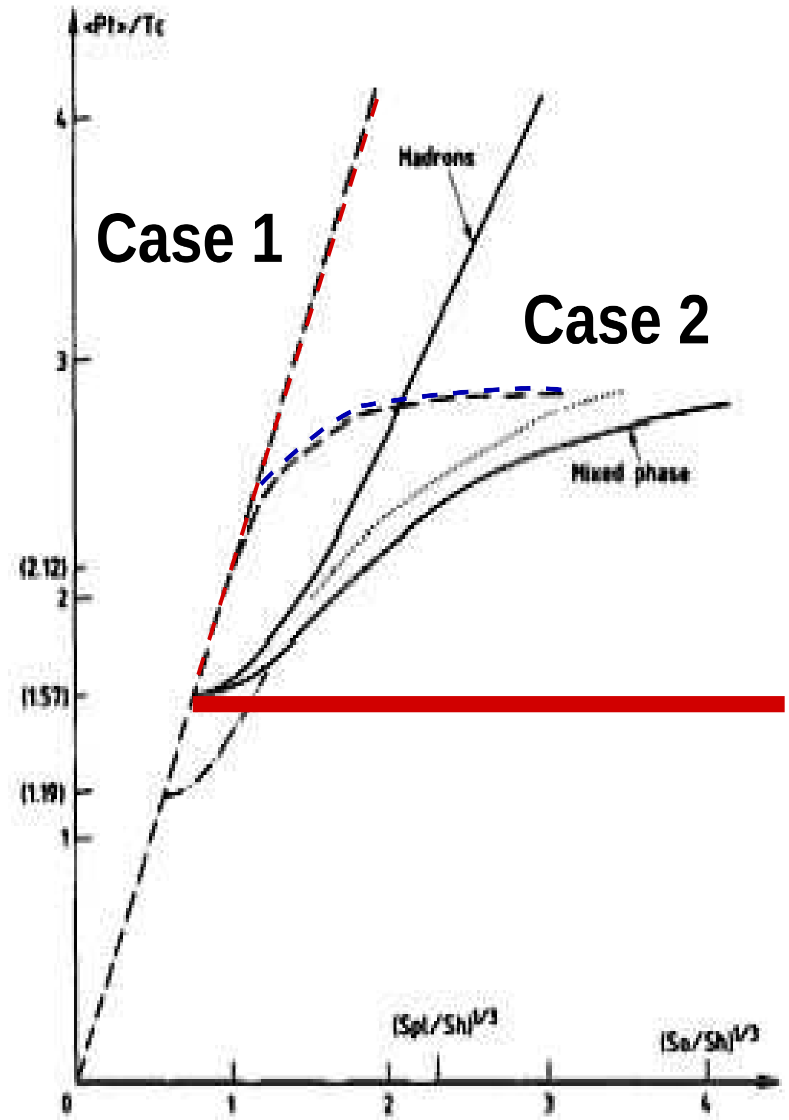
$$\underbrace{\langle p_T \rangle \simeq 2.82\, T_c}_{\text{Constant}}$$



<Pt>/Tc
Case 1
Case 2
Hadrons
Mixed phase
(2.12)
(1.57)
(1.19)
(S/Sh)**1/3

Longitudinal Expansion Only

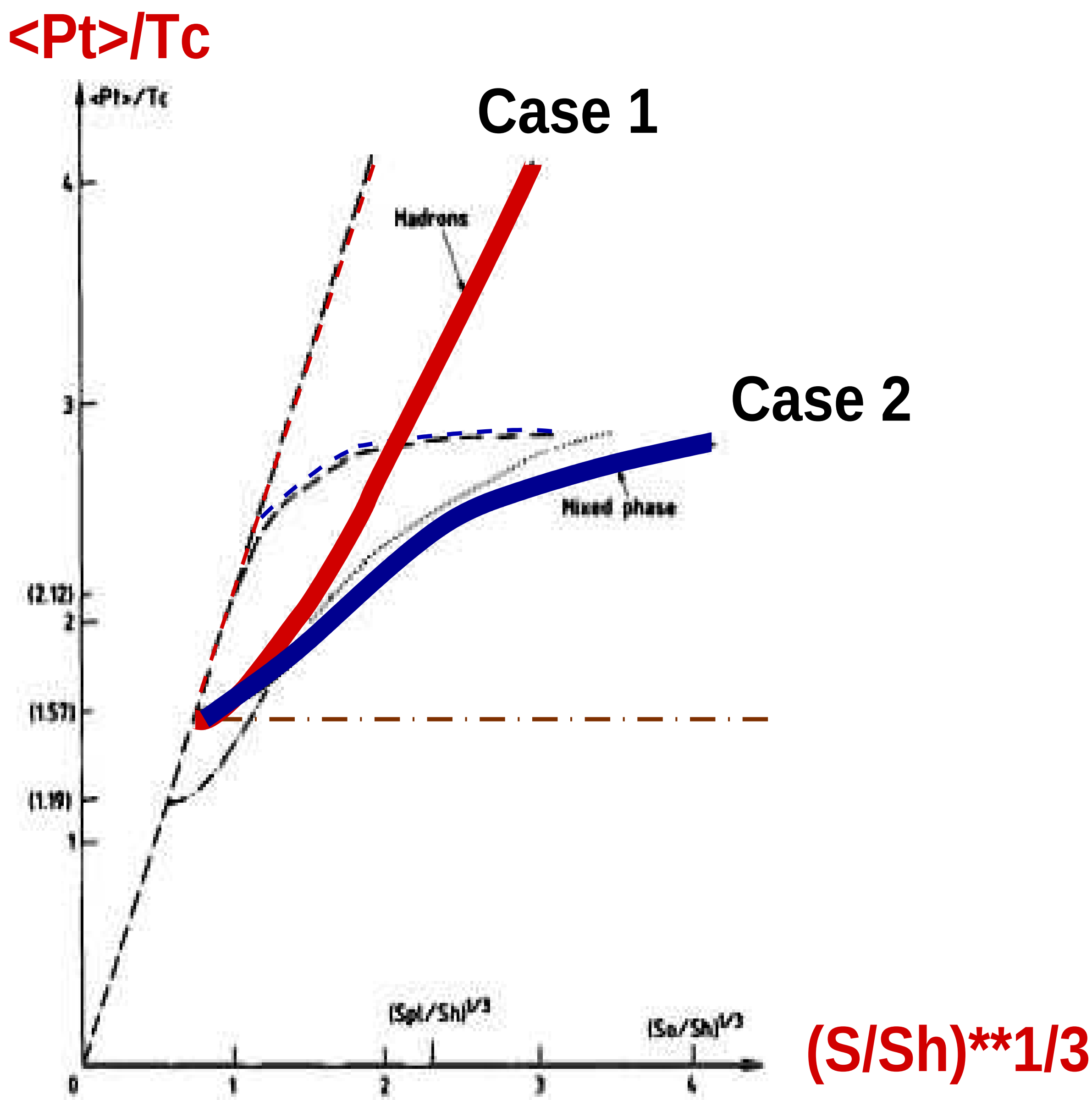
<Pt>/Tc



Pure Bjorken Freezeout at Constant T = 0.75 Tc

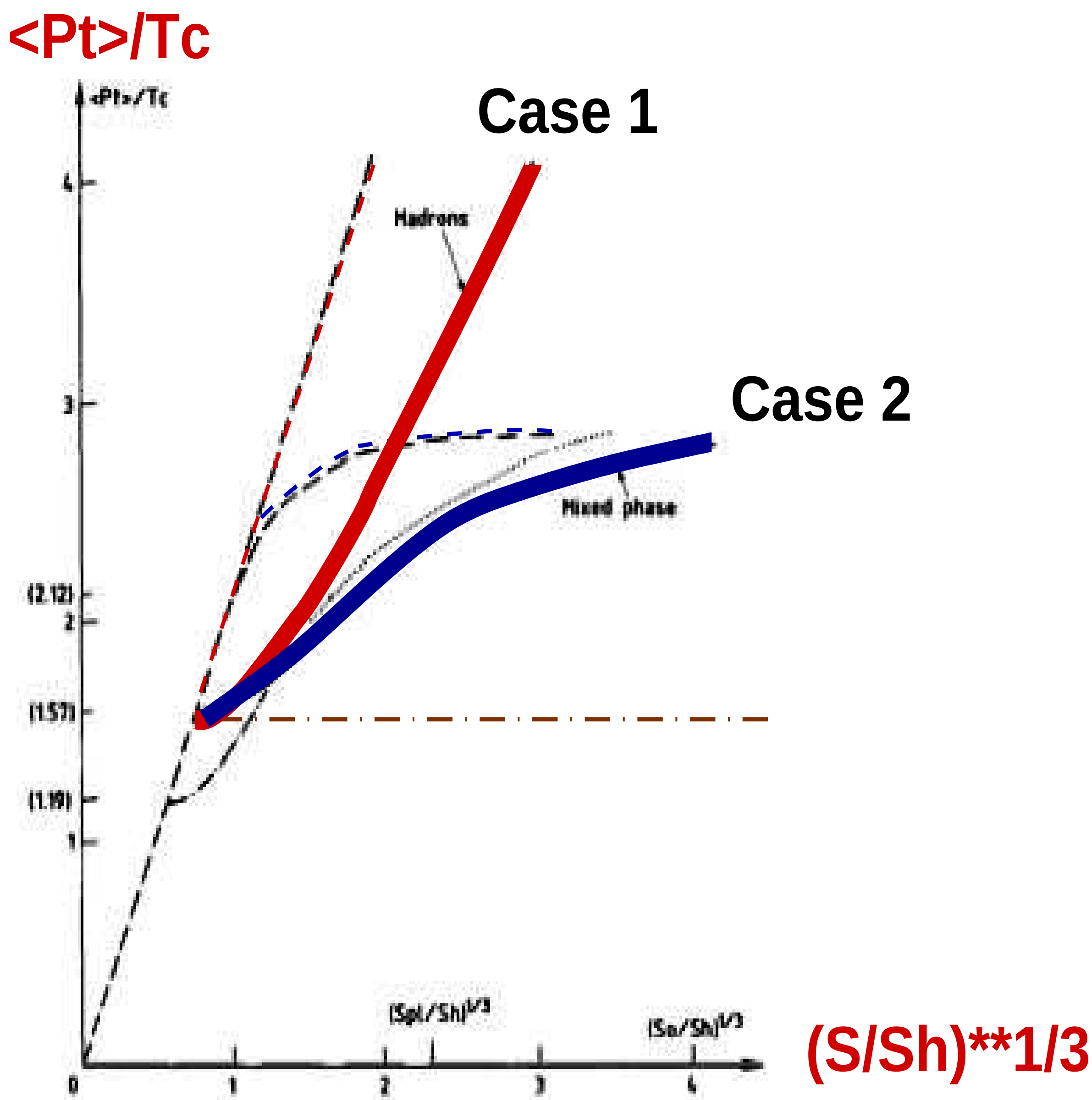
(S/Sh)1/3**

Hydro



<Pt>/Tc
<Pt>/Tc
Case 1
Case 2
Hadrons
Mixed phase
4
3
(2.12)
2
(1.57)
(1.19)
1
0
1
2
3
4
(Spl/Sh)^1/3
(So/Sh)^1/3
(S/Sh)**1/3

Hydro



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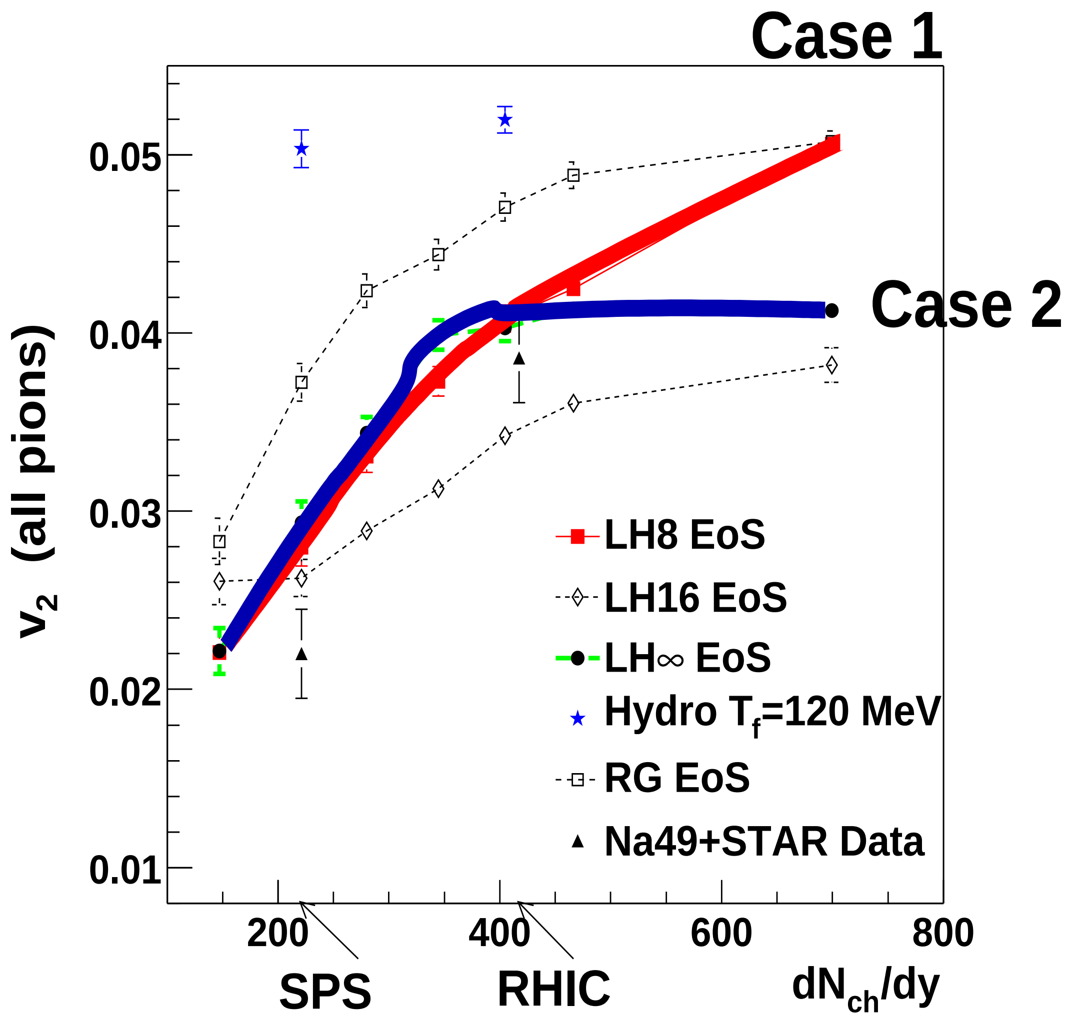
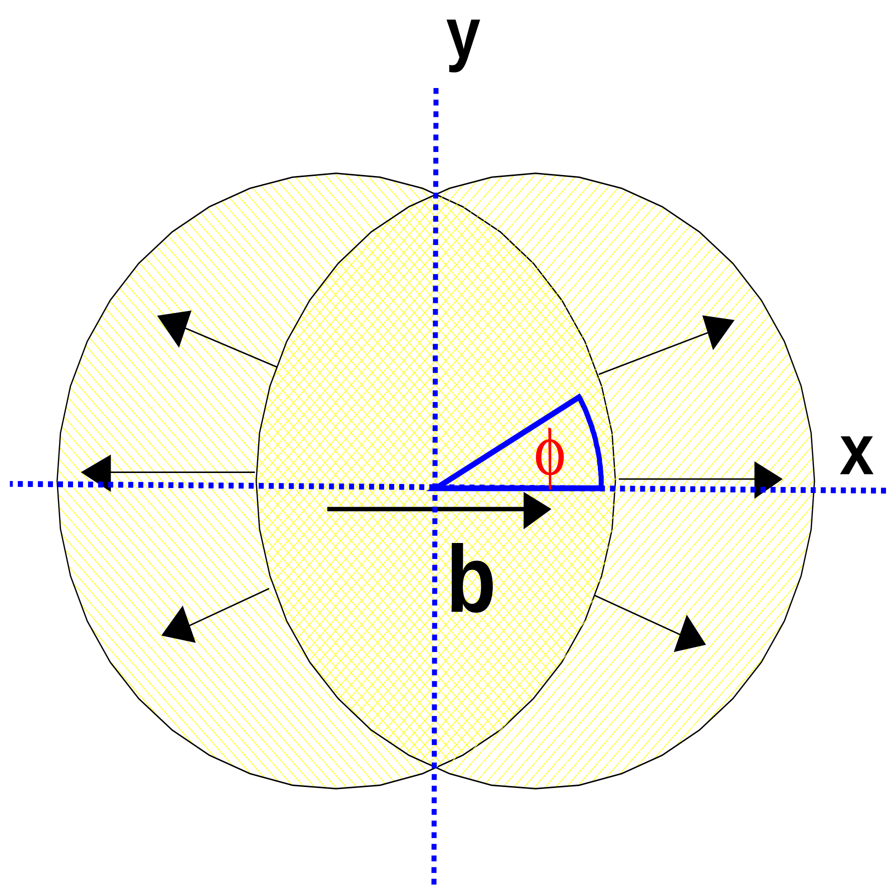
(So/Sh)^1/3
(S/Sh)**1/3

v_2 vs. Energy



No “Hydro Limit”

Hydro



Is the system *Large* enough? Does it live *Long* enough for hydro?

How *Long* and *Large* is Long/Large Enough ?

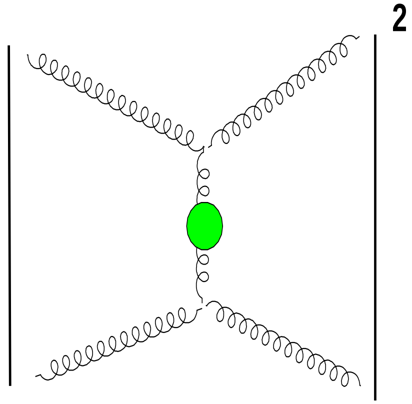
- Need the mean free path times expansion rate less than one

$$\ell_{\text{m.f.p.}} \times \text{Expansion Rate} \ll 1$$

How *Long* and *Large* is Enough ?

- Quick estimate of the mean free path:

$$\ell_{\text{m.f.p}} \equiv \frac{1}{n\sigma} = \frac{1}{\underbrace{n}_{\sim T^3} \times \underbrace{\sigma}_{\alpha_s^2/T^2}} \sim \frac{1}{\alpha_s^2 T}$$



- So the Figure of Merit:

$$\overbrace{\ell_{\text{m.f.p.}}}^{1/\alpha_s^2 T} \times \overbrace{\text{expansion rate}}^{1/\tau} \ll 1$$

$$\underbrace{\frac{1}{\alpha_s^2}}_{\text{Liquid Parameter}} \times \underbrace{\frac{1}{\tau T}}_{\text{Experimental Parameter}} \ll 1$$

How *Long* and *Large* is *Long/Large* Enough ?

- What is the mean free path? $\ell_{mfp} \equiv \frac{\eta}{e+p}$
- The mean free path should be less than the expansion rate $\frac{1}{\tau}$:

$$\underbrace{\frac{\eta}{e+p}}_{\ell_{mfp}} \frac{1}{\tau} \quad \ll \quad 1$$

- Then using the relation: $(e+p) = sT$.

$$\underbrace{\frac{\eta}{s}}_{\text{Liquid parameter}} \quad \times \quad \underbrace{\frac{1}{\tau T}}_{\text{Experimental parameter: } \sim 1} \quad \ll 1$$

1. η/s needs to be small to have interacting QGP at RHIC.
2. Even if η/s is small, dissipative effects are significant!

Extreme Estimate of η/s for the initial stage of the QGP

- Strongly Coupled conformal N=4 SYM – AdS/CFT Son, Starinets, Policastro

$$\left(\frac{\ell_{mfp}}{\tau}\right) = \underbrace{\frac{1}{4\pi}}_{\eta/s} \times \underbrace{\frac{1}{\tau T}}_{\sim 1}$$

Only with these numbers expect some collectivity.

Energy Dependence of the Shear Viscosity:

$$\underbrace{\frac{\eta}{s}}_{\text{Liquid parameter } \sim 1/\alpha_s^2} \quad \times \quad \underbrace{\frac{1}{\tau T}}_{\text{Experimental parameter}} \quad \ll 1$$

- η/s is a slow function of energy.
- η/s at the SPS is roughly the same as at RHIC
- RHIC shows hydrodynamic response because of the

Experimental Parameter

LHC will be an even better liquid.

Time Dependence of the Shear Viscosity

- Summary at time τ_0

$$T_o \sim 300\,\text{MeV} \quad \text{and} \quad \tau_0 \sim 1\,\text{fm}$$

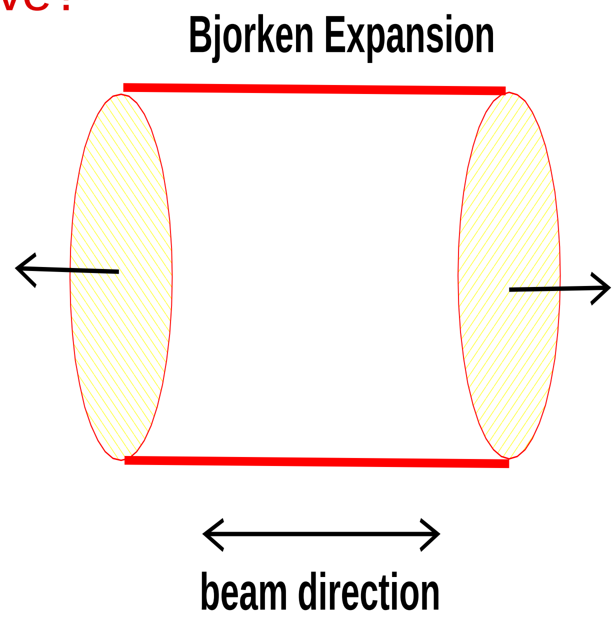
- Find:

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.1 - 0.4$$

How does $\frac{\Gamma_s}{\tau}$ evolve?

- 1D Expansion – scales set by temperature.
- 3D Expansion – scales fixed.

How does Γ_s/τ evolve?

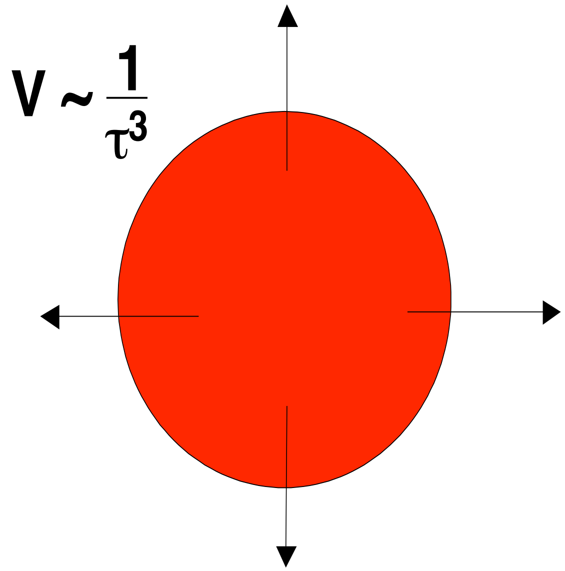


- 1D Bjorken Expansion – scales set by temperature
  - Temperature decreases $T \sim \frac{1}{\tau^{1/3}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{2/3}}$$

Viscous effects get steadily smaller

How does Γ_s/τ evolve?



- 3D Expansion – scales fixed
  - Density decreases $n \sim \frac{1}{\tau^3}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau n \sigma_o} \sim \# \frac{\tau^2}{\sigma_o}$$

Viscous effects get rapidly larger

Solving the Relativistic Navier Stokes Equations RNSE

- The RNSE as written can not be solved. There are unstable modes which propagate faster than the speed of light.

- Why? Because the stress RNSE tensor is not allowed time to change.

$$T^{ij}_{vis}\Big|_{\text{instantly}} = \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i \right)$$

- Can make many models which relax to the RNSE.

$$T^{ij}_{vis}\Big|_{\omega\to 0} \sim \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i \right)$$

- In the regime of validity of hydrodynamics the models all agree with each other and with RNSE.

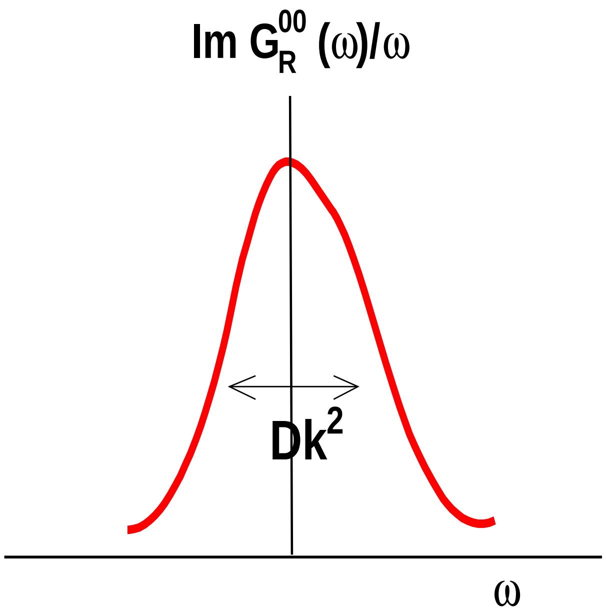
Can solve these models

Diffusion Equation

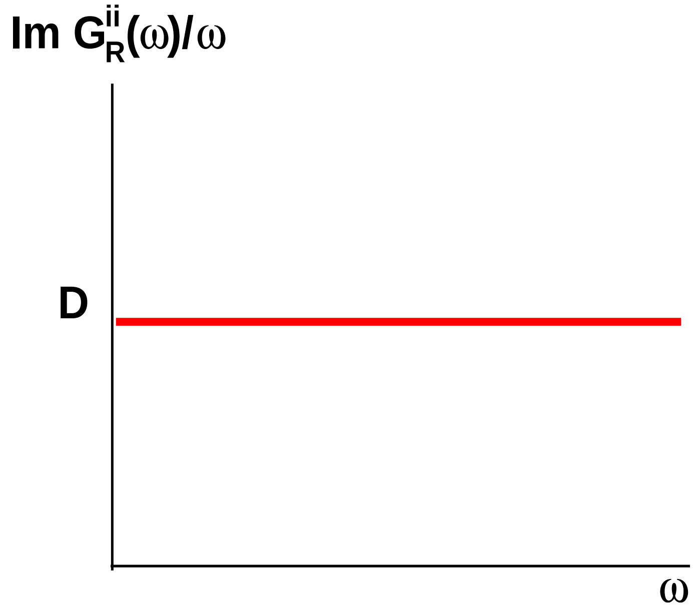
$$\partial_t n - D\nabla^2 n = 0$$

- Specifies the form of the spectral density at small k and ω

$$G_R(\omega, k) = \frac{1}{\partial_t - D\nabla^2} = \frac{1}{-i\omega + Dk^2}$$



$\Longrightarrow$

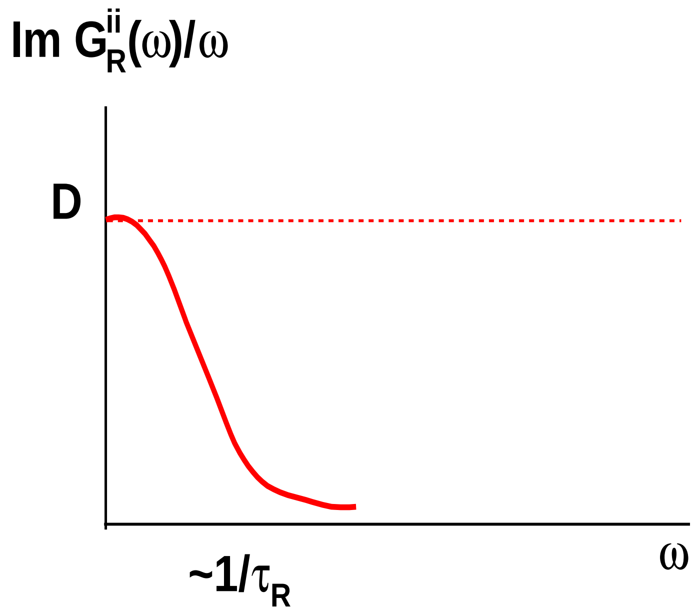


Relaxation Time Approximation:

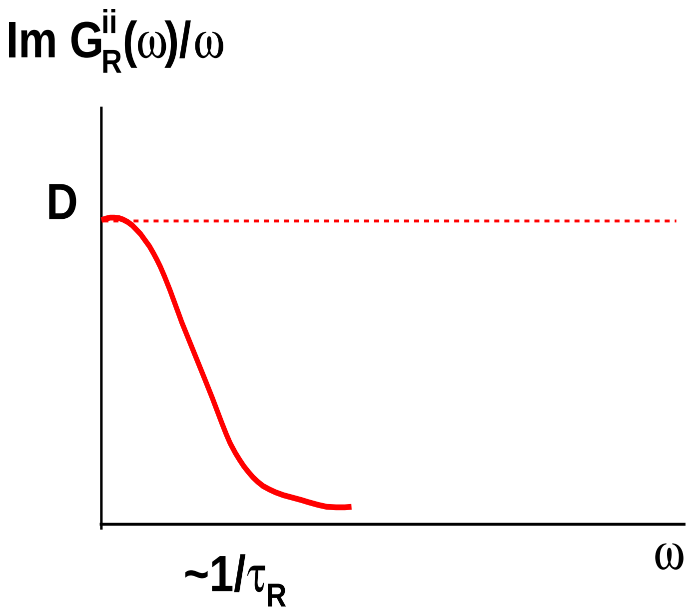
$$
\begin{aligned}
\partial_t n + \partial_x j &= 0 \\
\partial_t j &= -\frac{(j + D\nabla n)}{\tau_R}
\end{aligned}
$$

- Solve the system equations and find the retarded correlator

$$
\frac{\mathrm{Im} G_R(\omega)}{\omega} = \frac{D}{\pi} \frac{1}{1 + (\omega \tau_R)^2}
$$



Weak Coupling Sum Rules and Short Time Response



- f-Sum Rule at Weak Coupling

$$\underbrace{\int d\omega \, \frac{\mathrm{Im} G_R^{ii}(\omega)}{\omega}}_{\text{Short Times}} = \left\langle v_p^2 \right\rangle$$

- Substitute the model $G_R(\omega)/\omega$

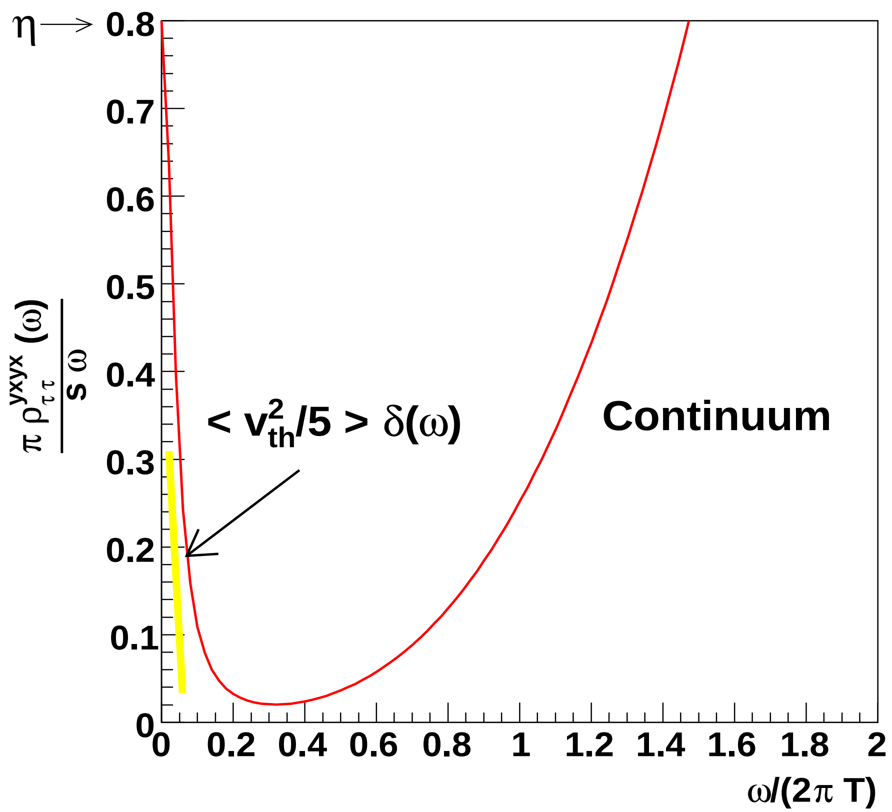
$$\frac{D}{\underbrace{\tau_R}_{\text{Short Times}}} = \left\langle v_p^2 \right\rangle$$

- Use short and long time parameters:
  - Long Time Parameters: D
  - Short Time Parameters: $\frac{D}{\tau_R} = \left\langle v_p^2 \right\rangle$

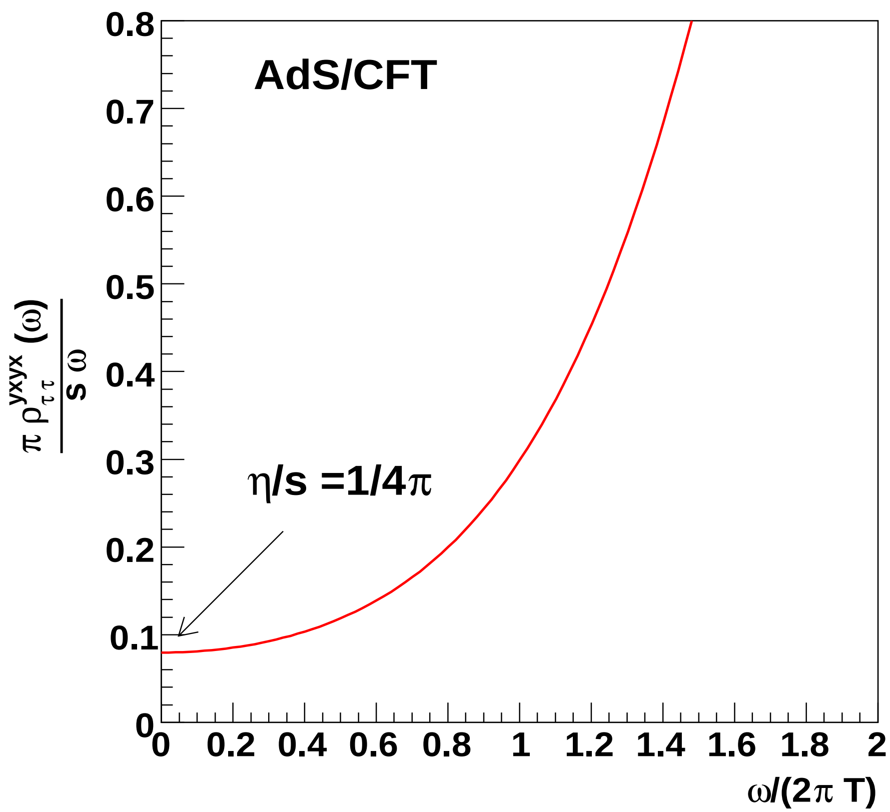
Real Spectral Densities:

- Relaxation models are a one parameter ansatz for the spectral density at small frequency which satisfy the f-Sum Rule

Cartoon of Weak Coupling



Strong Coupling (Kovtun, Starinets; DT)



A Lorentzian ansatz may be a poor choice.

Relaxation Time Approximation – Bjorken Expansion

1. Normal Viscous Hydro

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \qquad T^{zz}_{eq} = p - \frac{4}{3}\frac{\eta}{\tau}$$

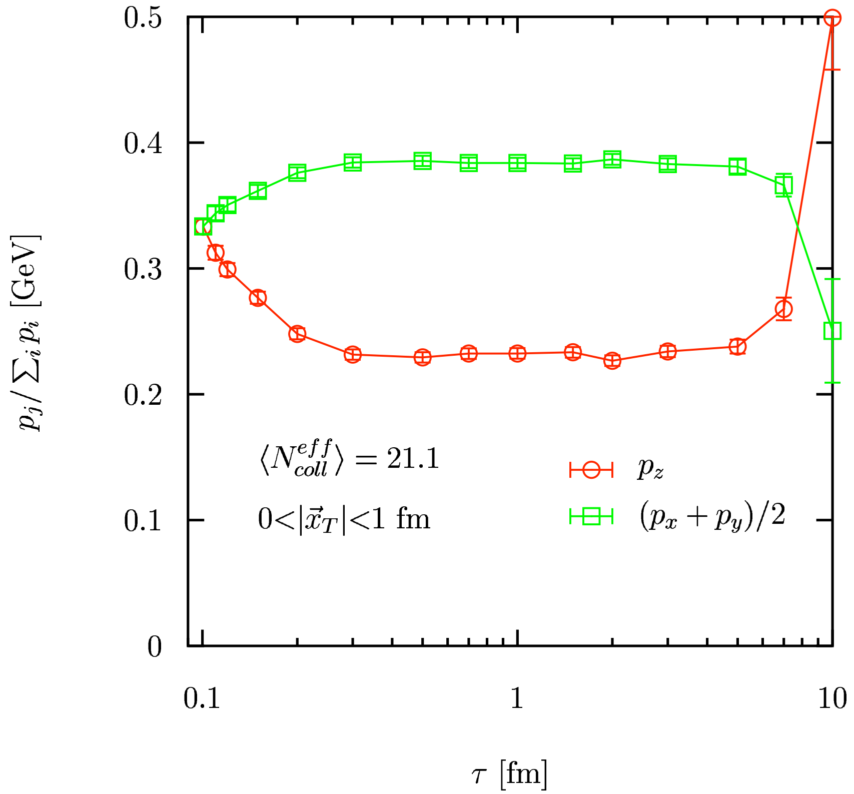
2. Relaxation Time Approximation

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \quad \text{and} \quad \frac{dT^{zz}}{d\tau} = -\frac{(T^{zz} - T^{zz}_{eq})}{\tau_R}$$

   - What are the appropriate initial conditions for this second equation?

Answer: $T^{zz} \simeq T^{zz}_{eq}$

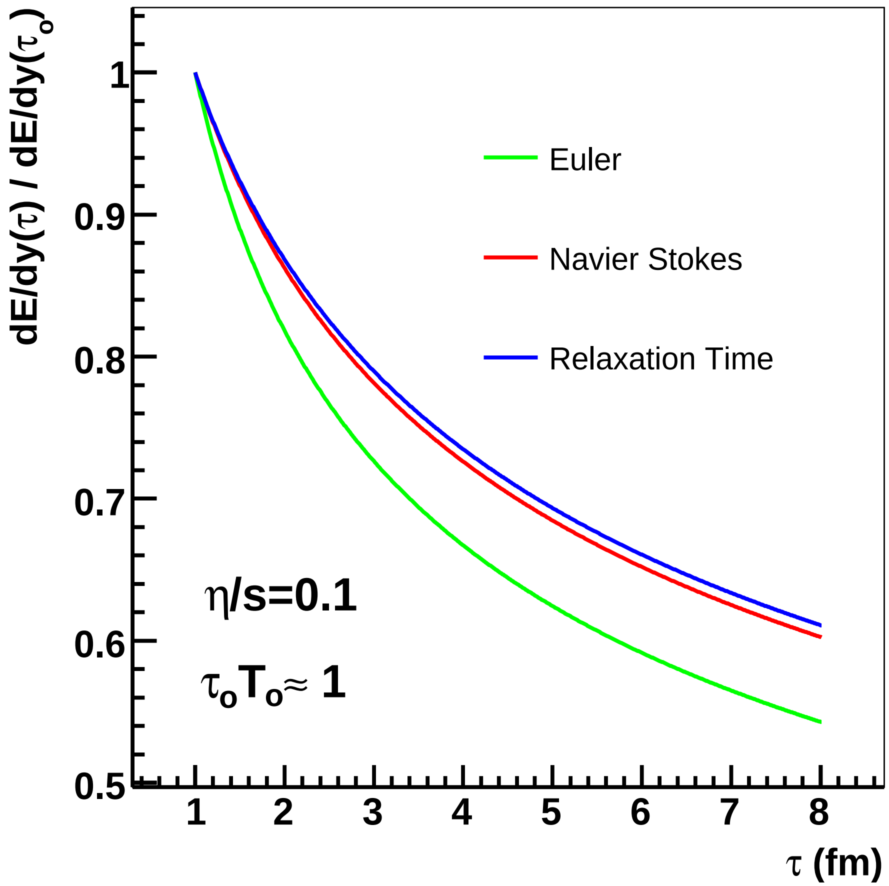
Kinetic Theory Calculations by D. Molnar



The stress tensor rapidly approaches quasi -stationary form

$$
\begin{aligned}
T^{zz} &= p - \frac{4}{3}\frac{\eta}{\tau} \\
T^{xx} &= p + \frac{2}{3}\frac{\eta}{\tau}
\end{aligned}
$$

Solution of Relaxation Time Equations



Relaxation is practically the same as Navier Stokes

Made precise – L. Lindblom

Relaxation Time Approximation – Bjorken Expansion

1. Normal Viscous Hydro

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \qquad T^{zz}_{eq} = p - \frac{4}{3}\frac{\eta}{\tau}$$

2. Relaxation Time Approximation

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \quad \text{and} \quad \frac{dT^{zz}}{d\tau} = -\frac{(T^{zz} - T^{zz}_{eq})}{\tau_R}$$

   – Two Models for the Relaxation Time Approximation

A simple model: Inspired by H.C. Ottinger, Physica 1997

- Imagine a tensor c_{ij} which relaxes quickly to $\partial_i v_j + \partial_j v_i$

$$\partial_t c_{ij} - (\partial_i v_j + \partial_j v_i) = \frac{\bar{c}_{ij}}{\tau_0} + \frac{\langle c_{ij} \rangle}{\tau_2}$$

where $\bar{c}_{ij} = (tr\,\mathbf{c})\,\delta_{ij}$ and $\langle c_{ij} \rangle = c_{ij} - \frac{1}{3}\bar{c}_{ij}$

- For small τ_0 and τ_2 we have:

$$c_{ij} \approx \tau_0 \delta_{ij}\, \partial_i v^i + \tau_2 (\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

- Then the “effective” pressure for small strains is given by:

$$T_{ij} \approx p(\delta_{ij} - a_1\, c_{ij})$$

Compare this to the canonical form:

$$T_{ij} \approx p\delta_{ij} + \sigma \partial_i v^i + \eta(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

Can map, $(\tau_0, \tau_2, a_1) \rightarrow (\sigma, \eta, c_\infty)$

Another Model: (Inspired by Lindblom and Geroch, Phys. Dev. D1994)

- Write a set conservation/balance laws:

$$
\begin{aligned}
\partial_\mu(N^\mu) &= 0 \\
\partial_\mu(T^{\mu\nu}) &= 0 \\
\partial_\mu(A^{\mu\alpha\beta}) &= I^{\alpha\beta}
\end{aligned}
$$

where

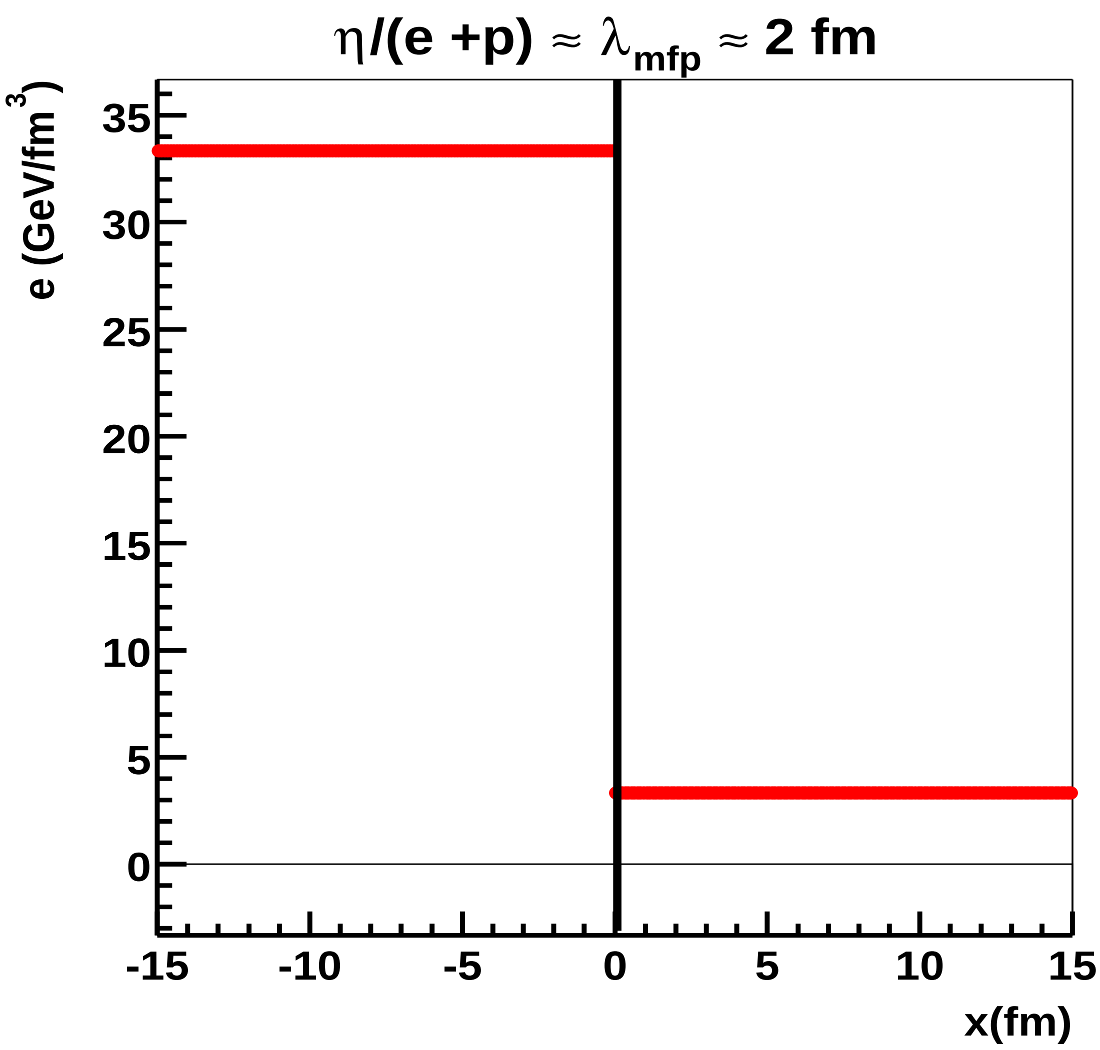
$$
\begin{aligned}
N^\mu &= n u^\mu \\
T^{\mu\nu} &= e u^\mu u^\nu + p\Delta^{\mu\nu} + u^\mu q^\nu + u^\nu q^\mu + \tau^{\mu\nu} \\
A^{\mu\alpha\beta} &= 2T\Delta^{\mu(\alpha}u^{\beta)} \\
I^{\alpha\beta} &= -\frac{T}{\eta}\tau^{\alpha\beta} - \frac{2T}{3\sigma}\Delta^{\alpha\beta} - \frac{2T}{\kappa T}\left(q^\alpha u^\beta + q^\beta u^\alpha\right)
\end{aligned}
$$

- A completely different model at short times

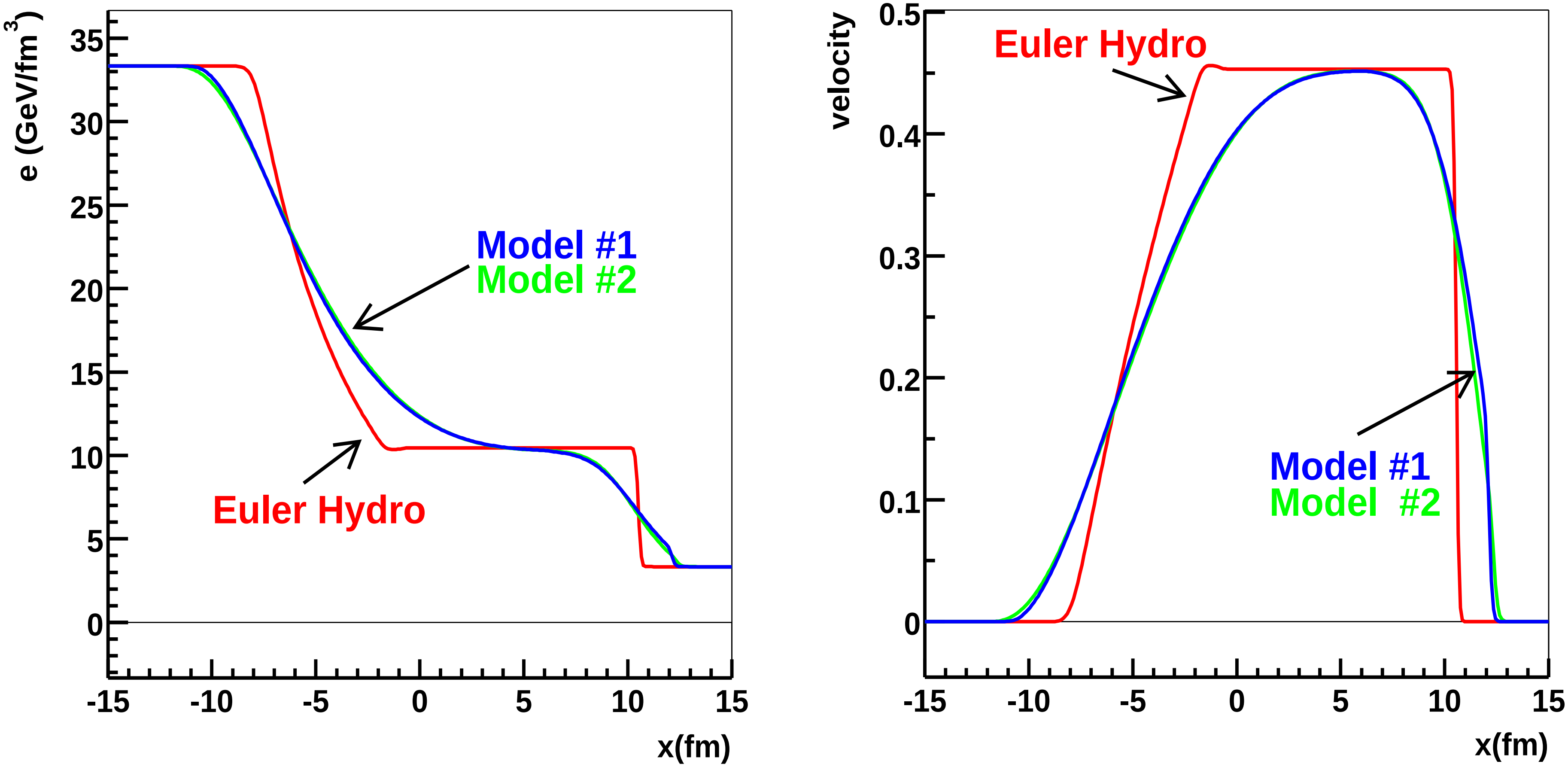
- Only the long time behavior is the same. The long time behavior is controlled by the viscous coefficients.

None of the details of these models should matter.

Sod’s Test Problem

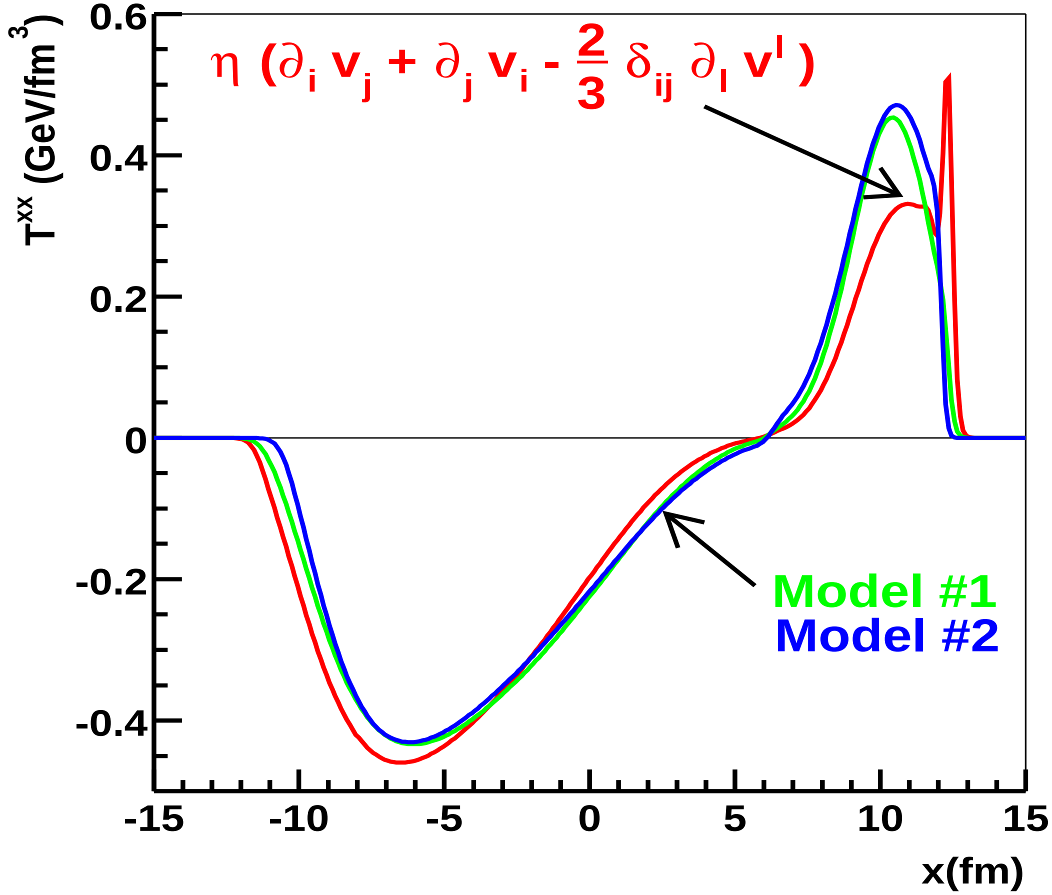


Compare the different models:



The solutions are very similar but different from ideal hydro.

Compare the stress tensor with the Navier Stokes Equations:



The stress tensor is close to its canoncial form.

Summary & Warnings

- All models agree about the solution to the Navier Stokes equations
- The stress energy tensor is almost always very close to

$$T^{ij} \sim \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3} \delta^{ij} \partial_l v^l \right)$$

This holds in the regime of validity of hydrodynamics.

1. The only natural initial condition is

$$T^{ij}|_{\tau_0} = \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3} \delta^{ij} \partial_l v^l \right)$$

2. In general the models have several free parameters. The solution only depends on the viscosity and not short time parameters.

Running Viscous Hydro in Three Steps

1. Run the evolution and monitor the viscous terms

2. When the viscous term is about half of the pressure:

   – The models disagree with each other.

   – T^{ij} is not asymptotic with $\sim \eta(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_l v^l)$

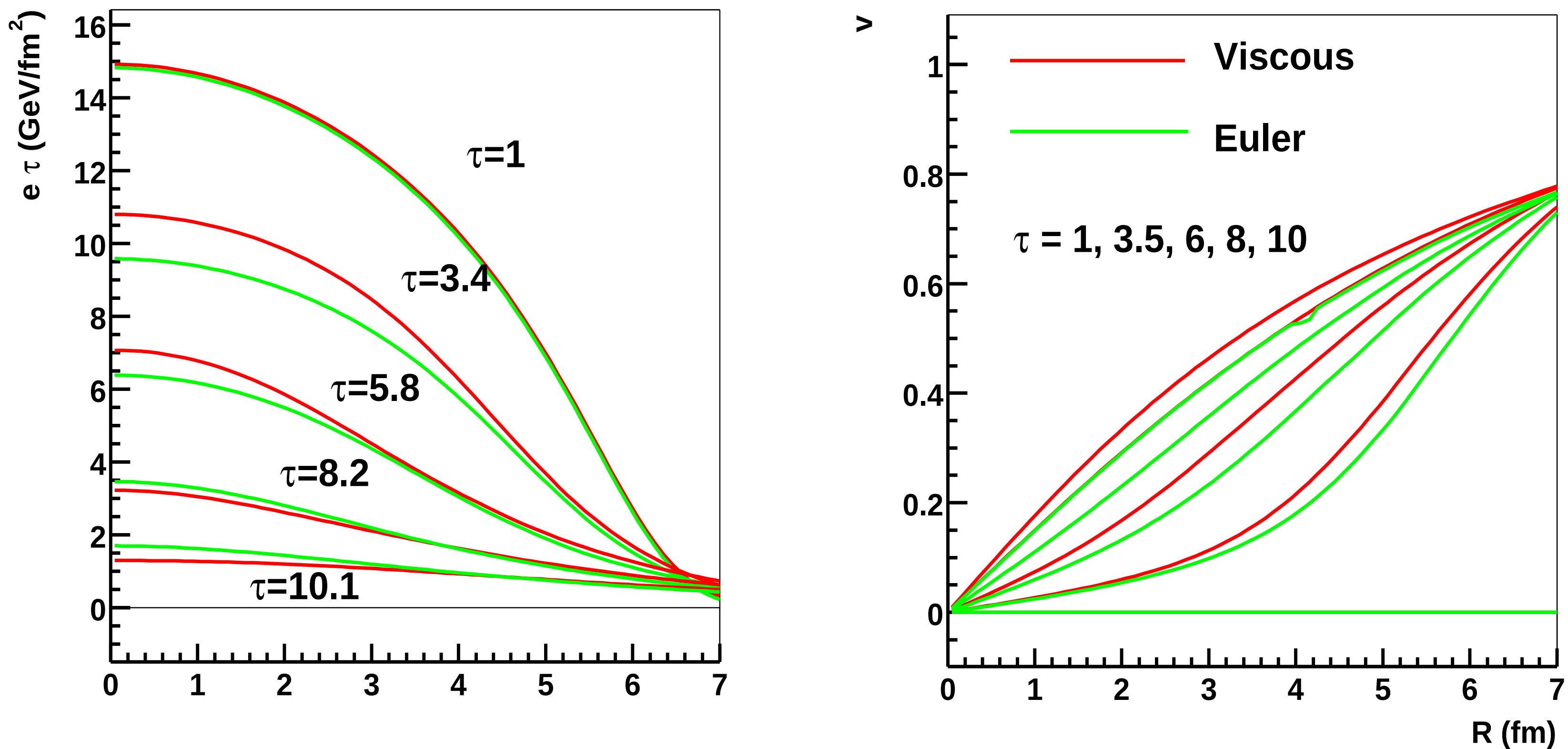
   Freezeout is signaled by the equations.

3. Compute spectra:

   – Viscous corrections to the spectra grow with p_T

   Maximum p_T is also signaled by the equations.

Bjorken Solution with transverse expansion: Step 1

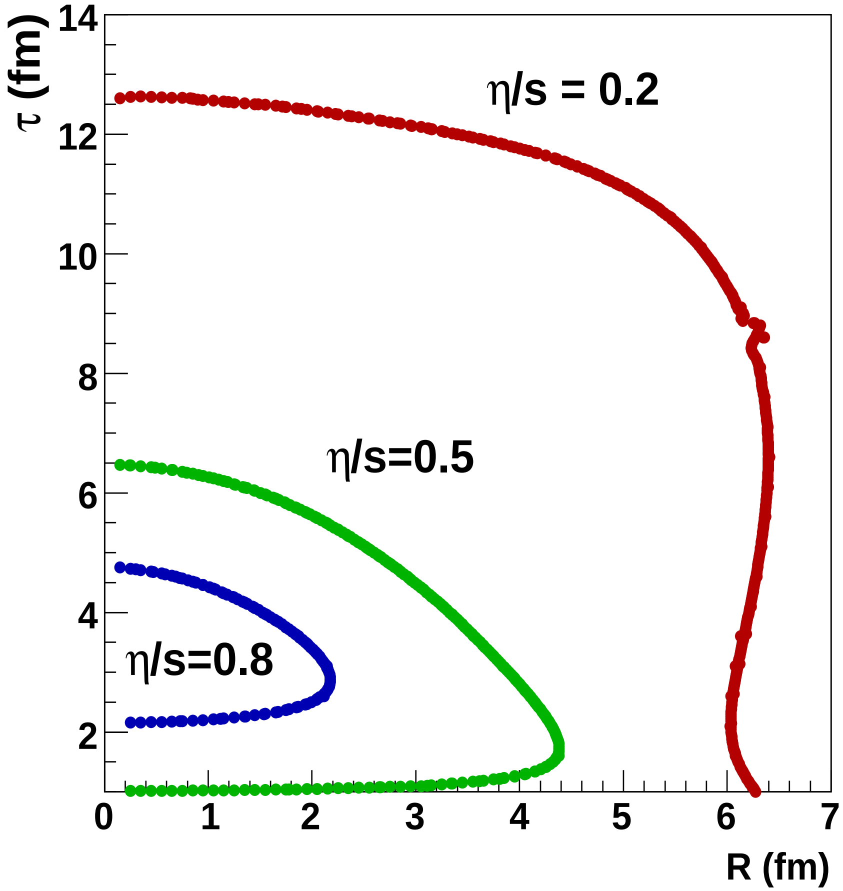


- First the viscous case does less longitudinal work.
- Then the transverse velocity grows more rapidly because the transverse pressure is larger.
- The larger transverse velocity then reduces the energy density more quickly than ideal hydro.

Viscous corrections do NOT integrate to give an O(1) change to the flow.

Monitor the viscous terms and compute freezeout: Step 2

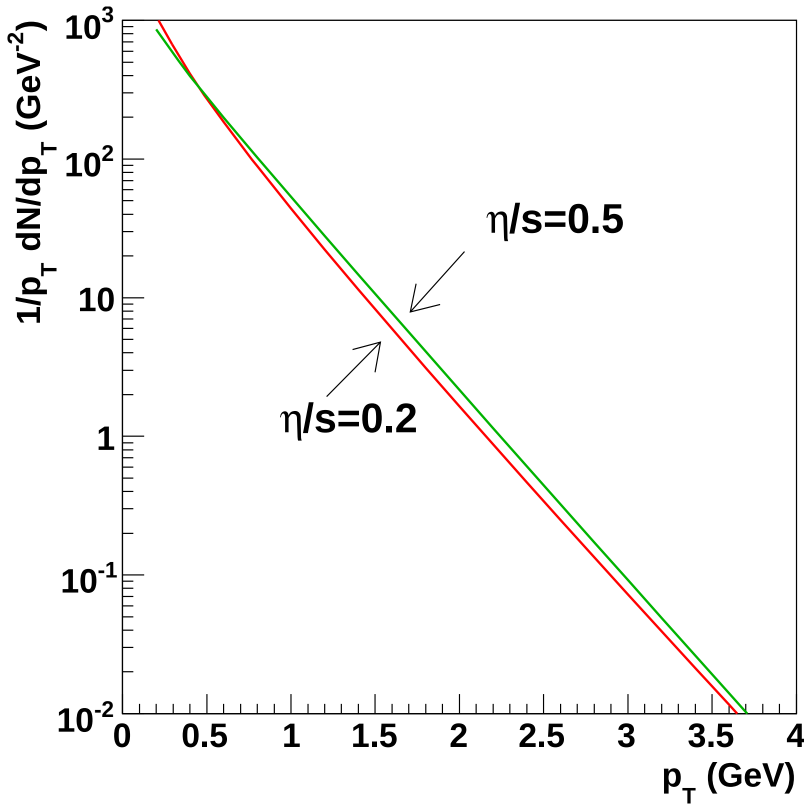
- Contours where viscous terms become O(1)

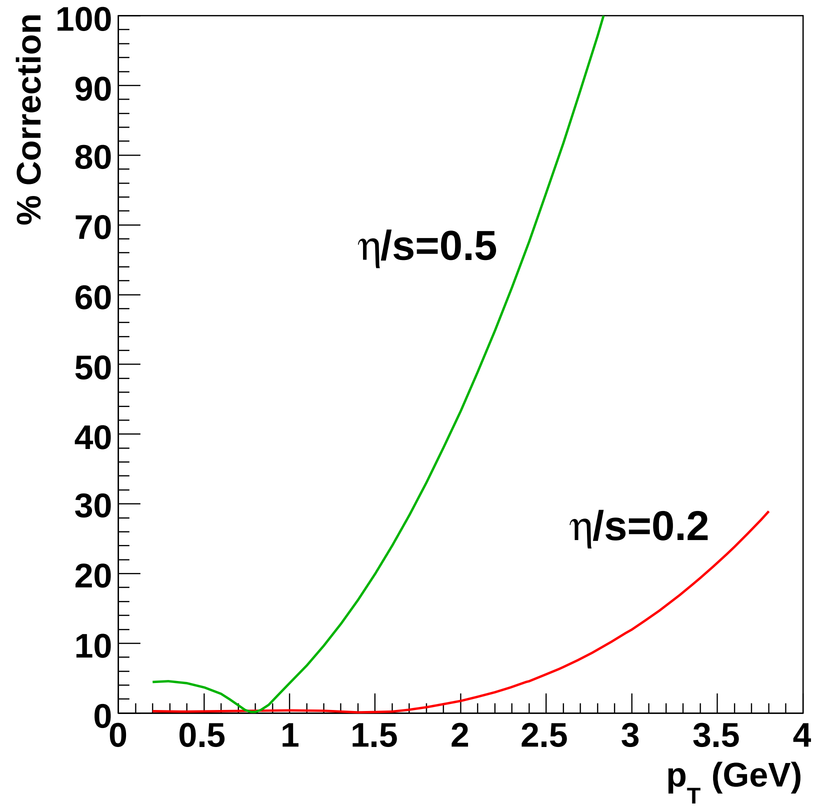


The space-time volume where hydro applies depends strongly on η/s

Compute the spectra with the viscous correction: Step 3

$$f \rightarrow f_o + p^i p^j \ \langle \partial_i v_j \rangle$$





Limitations:

- Viscous hydro resums terms which grow with time.

$$\text{Time} \times \frac{\ell_{\text{m.f.p}}}{L^2}$$

  - If the system has finite lifetime these effects are unimportant.
  - Important for the Sod Problem but not for Heavy Ions

- Other “viscous” effects are more important and not included
  - Finite Opacity – Particles Escape
  - Memory Effects
  - Important for Heavy Ions but not for the Sod Problem

Conclusions:

- Viscosity does not change the ideal hydrodynamic solution much. Time is not too long.
- "Viscous" effects are very important
- It signals the boundary of applicability
- Constrain Viscosity/Opacity/Mean Free Path from other observables